Linear Algebra II

05/04/2016, Tuesday, 9:00 – 12:00

1 $(3 + (3 + 6 + 3) = 15$ pts) **Gram-Schmidt process**

Consider the vector space $\mathbb{R}^5$ with the inner product $\langle x, y\rangle = x^T y$.

1. Consider the vectors

$$x_1 = \begin{bmatrix}1 & 1 & 0 & 1 & 1\end{bmatrix}^T, \quad x_2 = \begin{bmatrix}1 & -1 & 1 & 0 & -1\end{bmatrix}^T, \quad x_3 = \begin{bmatrix}2 & 2 & 1 & -4 & 0\end{bmatrix}^T.$$

Let θ_{ij} denote the angle between x_i and x_j. Find $\cos\theta_{12}$, $\cos\theta_{23}$, and $\cos\theta_{31}$.

2. Let

$$A = \begin{bmatrix}1 & -1 & 0 & 0 & 1\\ 0 & 1 & -1 & 0 & 1\end{bmatrix}.$$

   (i) Find a basis for $N(A)$, the null space of A.

   (ii) Apply Gram-Schmidt process to obtain an orthonormal basis for $N(A)$.

   (iii) Find the closest element of $N(A)$ to the vector $\begin{bmatrix}1 & 1 & 1 & 1 & 1\end{bmatrix}^T$.

Required Knowledge: **inner product, Gram-Schmidt process, least squares**

Solution:

(1a): Note that

$$\begin{aligned}\langle x_1, x_1\rangle &= 4\\ \langle x_1, x_2\rangle &= -1\\ \langle x_1, x_3\rangle &= 0\\ \langle x_2, x_2\rangle &= 4\\ \langle x_2, x_3\rangle &= 1\\ \langle x_3, x_3\rangle &= 25.\end{aligned}$$

Then, we get

$$\cos\theta_{12} = \frac{\langle x_1, x_2\rangle}{\|x_1\|\|x_2\|} = -\frac{1}{4} \qquad \cos\theta_{23} = \frac{\langle x_2, x_3\rangle}{\|x_2\|\|x_3\|} = \frac{1}{10} \qquad \cos\theta_{31} = \frac{\langle x_3, x_1\rangle}{\|x_3\|\|x_1\|} = 0.$$

(1b):

(i): The null space of A is given by $N(A) = \{x \mid Ax = 0\}$. Note that

$$\begin{bmatrix}1 & -1 & 0 & 0 & 1\\ 0 & 1 & -1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = 0$$

leads to

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}x_3 - 2x_5\\ x_3 - x_5\\ x_3\\ x_4\\ x_5\end{bmatrix} = x_3\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix} + x_4\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} + x_5\begin{bmatrix}-2\\-1\\0\\0\\1\end{bmatrix}.$$

This means that

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $N(A)$.

(ii): Note that

$$\begin{aligned} \langle x_1, x_1 \rangle &= 3 \\ \langle x_1, x_2 \rangle &= 0 \\ \langle x_1, x_3 \rangle &= -3 \\ \langle x_2, x_2 \rangle &= 1 \\ \langle x_2, x_3 \rangle &= 0 \\ \langle x_3, x_3 \rangle &= 6. \end{aligned}$$

where

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad x_3 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

By applying the Gram-Schmidt process, we obtain:

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$u_2 = \frac{x_2 - p_1}{\|x_2 - p_1\|} \qquad p_1 = \langle x_2, u_1 \rangle \cdot u_1$$

$$= 0$$

$$x_2 - p_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\|x_2 - p_1\|^2 = 1$$

$$u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$u_3 = \frac{x_3 - p_2}{\|x_3 - p_2\|} \qquad p_2 = \langle x_3, u_1 \rangle \cdot u_1 + \langle x_3, u_2 \rangle \cdot u_2$$

$$= \frac{1}{3} \cdot -3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$x_3 - p_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\|x_3 - p_2\|^2 = 3$$

$$u_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

(iii): The closest element in $N(A)$ to $x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ can be found by projection:

$$p = \langle x, u_1 \rangle \cdot u_1 + \langle x, u_2 \rangle \cdot u_2 + \langle x, u_3 \rangle \cdot u_3.$$

Thus, we have

$$p = \frac{1}{3} \cdot 3 \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ \frac{4}{3} \\ 1 \\ \frac{1}{3} \end{bmatrix}.$$

Consider the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$$

where a, b, and c are real numbers.

a. Determine all values of (a, b, c) such that M is unitarily diagonalizable.

b. Find a unitary diagonalizer of M for each triple (a, b, c) found in (a).

REQUIRED KNOWLEDGE: **unitarily diagonalization**

SOLUTION:

(2a): A matrix is unitarily diagonalizable if and only if it is normal. Then, we need to find conditions on (a, b, c) such that

$$M^T M = M M^T.$$

Note that

$$M^T M = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix} = \begin{bmatrix} a^2 & ab & ac \\ ab & 1+b^2 & bc \\ ac & bc & 1+c^2 \end{bmatrix}$$

and

$$M M^T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix} \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & c \\ b & c & a^2+b^2+c^2 \end{bmatrix}$$

By looking at the diagonals, we obtain $a^2 = 1$ and $b = 0$. By looking at the 31-elements, we see that $c = 0$. These choices result in $M^T M = M M^T$. Therefore, M is unitarily diagonalizable if and only if

$$a^2 = 1 \qquad b = 0 \qquad c = 0.$$

(2b): For the two possibilities found in (a), we have

$$M_- = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad M_+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

In order to diagonalize M_-, note that

$$p_-(\lambda) = \det(M_- - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & 0 & -\lambda \end{bmatrix}\right) = -(\lambda^3 + 1) = -(\lambda+1)(\lambda^2 - \lambda + 1).$$

This results in the following eigenvalues:

$$\lambda_1 = -1 \qquad \lambda_{2,3} = \frac{1 \pm \sqrt{3}i}{2}.$$

Note that

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -1 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} -\lambda \\ -\lambda^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda^3 + 1 \\ 0 \end{bmatrix}.$$

Therefore, if λ is an eigenvalue of M_- then

$$\begin{bmatrix} -\lambda \\ -\lambda^2 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to λ. This leads to

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} \frac{-1-\sqrt{3}i}{2} \\ \frac{1-\sqrt{3}i}{2} \\ 1 \end{bmatrix} \qquad x_3 = \begin{bmatrix} \frac{-1+\sqrt{3}i}{2} \\ \frac{1+\sqrt{3}i}{2} \\ 1 \end{bmatrix}$$

for eigenvalues, respectively, λ_1, λ_2, and λ_3. Note that

$$\|x_1\| = \|x_2\| = \|x_3\| = 3.$$

Then, the unitary diagonalizer can be given by

$$U_- = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \frac{-1-\sqrt{3}i}{2} & \frac{-1+\sqrt{3}i}{2} \\ -1 & \frac{1-\sqrt{3}i}{2} & \frac{1+\sqrt{3}i}{2} \\ 1 & 1 & 1 \end{bmatrix}.$$

Indeed, it can be verified that

$$\begin{aligned} M_-U_- &= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \frac{-1-\sqrt{3}i}{2} & \frac{-1+\sqrt{3}i}{2} \\ -1 & \frac{1-\sqrt{3}i}{2} & \frac{1+\sqrt{3}i}{2} \\ 1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \frac{-1-\sqrt{3}i}{2} & \frac{-1+\sqrt{3}i}{2} \\ -1 & \frac{1-\sqrt{3}i}{2} & \frac{1+\sqrt{3}i}{2} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & \frac{1+\sqrt{3}i}{2} & \\ & & \frac{1-\sqrt{3}i}{2} \end{bmatrix} = U_-D_-. \end{aligned}$$

In order to diagonalize M_+, note that

$$p_+(\lambda) = \det(M_+ - \lambda I) = \det(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix}) = -(\lambda^3 - 1) = -(\lambda - 1)(\lambda^2 + \lambda + 1).$$

This results in the following eigenvalues:

$$\lambda_1 = 1 \qquad \lambda_{2,3} = \frac{-1 \pm \sqrt{3}i}{2}.$$

Note that

$$\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix} \begin{bmatrix} \lambda \\ \lambda^2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\lambda^3 + 1 \\ 0 \end{bmatrix}.$$

Therefore, if λ is an eigenvalue of M_+ then

$$\begin{bmatrix} \lambda \\ \lambda^2 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to λ. This leads to

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} \dfrac{-1+\sqrt{3}i}{2} \\ \dfrac{-1-\sqrt{3}i}{2} \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} \dfrac{-1-\sqrt{3}i}{2} \\ \dfrac{-1+\sqrt{3}i}{2} \\ 1 \end{bmatrix}$$

for eigenvalues, respectively, λ_1, λ_2, and λ_3. Note that

$$\|x_1\| = \|x_2\| = \|x_3\| = 3.$$

Then, the unitary diagonalizer can be given by

$$U_+ = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \dfrac{-1+\sqrt{3}i}{2} & \dfrac{-1-\sqrt{3}i}{2} \\ 1 & \dfrac{-1-\sqrt{3}i}{2} & \dfrac{-1+\sqrt{3}i}{2} \\ 1 & 1 & 1 \end{bmatrix}.$$

Indeed, it can be verified that

$$\begin{aligned} M_+U_+ &= \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & \dfrac{-1+\sqrt{3}i}{2} & \dfrac{-1-\sqrt{3}i}{2} \\ 1 & \dfrac{-1-\sqrt{3}i}{2} & \dfrac{-1+\sqrt{3}i}{2} \\ 1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & \dfrac{-1+\sqrt{3}i}{2} & \dfrac{-1-\sqrt{3}i}{2} \\ 1 & \dfrac{-1-\sqrt{3}i}{2} & \dfrac{-1+\sqrt{3}i}{2} \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & \dfrac{-1+\sqrt{3}i}{2} & \\ & & \dfrac{-1-\sqrt{3}i}{2} \end{bmatrix} = U_+D_+. \end{aligned}$$

Let $A \in \mathbb{R}^{n\times n}$ be of the companion form

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}.$$

a. Show that $A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I = 0$.

b. Show that if λ is an eigenvalue of A then $\begin{bmatrix}1 & \lambda & \lambda^2 & \cdots & \lambda^{n-2} & \lambda^{n-1}\end{bmatrix}^T$ is an eigenvector corresponding to λ.

Required Knowledge: **Eigenvalues, eigenvectors, Cayley-Hamilton theorem**

Solution:

(3a): First, note that the characteristic polynomial of the matrix A is given by

$$p_A(\lambda) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_2\lambda^2 + a_1\lambda + a_0)$$

since it is in the so-called companion form. Then, it follows from the Cayley-Hamilton theorem that

$$p_A(A) = 0,$$

equivalently

$$A^n + a_{n-1}A^{n-1} + \cdots + a_2A^2 + a_1A + a_0I = 0.$$

(3b): Note that

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \\ \vdots \\ \lambda^{n-1} \\ -a_0 - a_1\lambda - \cdots - a_{n-2}\lambda^{n-2} - a_{n-1}\lambda^{n-1} \end{bmatrix}.$$

Since λ is an eigenvalue, we have that $p_A(\lambda) = 0$. This leads to

$$\lambda^n = -a_{n-1}\lambda^{n-1} - \cdots - a_2\lambda^2 - a_1\lambda - a_0.$$

Then, we have

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \\ \vdots \\ \lambda^{n-1} \\ \lambda^n \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-2} \\ \lambda^{n-1} \end{bmatrix}.$$

Consequently, $\begin{bmatrix}1 & \lambda & \lambda^2 & \cdots & \lambda^{n-2} & \lambda^{n-1}\end{bmatrix}^T$ is an eigenvector corresponding to λ.

Consider the function

$$f(x, y, z) = ax^2 + y^2 + z^2 - xy + yz - xz$$

where $a \neq \frac{1}{3}$ is a real number.

a. Find all stationary points of f.

b. Determine all values of a such that the stationary point(s) are local minimum.

REQUIRED KNOWLEDGE: **stanionary points, local minima, positive definite matrices**

SOLUTION:

(4a): A stationary point $(\bar{x}, \bar{y}, \bar{z})$ satisfies

$$\begin{aligned} 0 &= 2a\bar{x} - \bar{y} - \bar{z} \\ 0 &= -\bar{x} + 2\bar{y} + \bar{z} \\ 0 &= -\bar{x} + \bar{y} + 2\bar{z}. \end{aligned}$$

Equivalently, it satisfies

$$\begin{bmatrix} 2a & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = 0.$$

Note that

$$\det(\begin{bmatrix} 2a & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}) = 8a + 1 + 1 - 2 - 2a - 2 = 6a - 2 \neq 0$$

since $a \neq \frac{1}{3}$. Therefore, the only solution to the above equation is the trivial solution $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 0)$. Thus, the only stationary point is the zero point.

(4b): We need to find out the Hessian matrix first:

$$H(x, y, z) = \begin{bmatrix} 2a & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Since this matrix does not depend on the variables, we have

$$H(0, 0, 0) = \begin{bmatrix} 2a & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

In order the stationary point $(0, 0, 0)$ to be a local minimum, $H(0, 0, 0)$ needs to be a positive definite matrix. This can be checked by employing the principal minor test:

$$\begin{aligned} \det(2a) &= 2a > 0 \\ \det(\begin{bmatrix} 2a & -1 \\ -1 & 2 \end{bmatrix}) &= 4a - 1 > 0 \\ \det(\begin{bmatrix} 2a & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}) &= 6a - 2 > 0. \end{aligned}$$

These inequalities, respectively, yield that $a > 0$, $a > \frac{1}{4}$, and $a > \frac{1}{3}$. Hence, $H(0, 0, 0)$ is positive definite if and only if $a > \frac{1}{3}$. Consequently, $(0, 0, 0)$ is a local minimum if $a > \frac{1}{3}$.

Let

$$M = \begin{bmatrix} a & -b & -c \\ a & -b & c \\ a & b & -c \\ a & b & c \end{bmatrix}$$

where a, b, and c real numbers with $a > b > c > 0$.

a. Find a singular value decomposition of M.

b. Find the best rank 2 approximation of M.

REQUIRED KNOWLEDGE: **singuar value decomposition, lower rank approximations**

SOLUTION:

(5a): Note that

$$M^T M = \begin{bmatrix} a & a & a & a \\ -b & -b & b & b \\ -c & c & -c & c \end{bmatrix} \begin{bmatrix} a & -b & -c \\ a & -b & c \\ a & b & -c \\ a & b & c \end{bmatrix} = \begin{bmatrix} 4a^2 & 0 & 0 \\ 0 & 4b^2 & 0 \\ 0 & 0 & 4c^2 \end{bmatrix}.$$

Since $a > b > c$, we can conclude that

$$\sigma_1 = 2a, \qquad \sigma_2 = 2b, \qquad \sigma_3 = 2c,$$

and $V = I$. This leads to

$$u_1 = \frac{1}{2a} A v_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad u_2 = \frac{1}{2b} A v_2 = \frac{1}{2} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \qquad u_3 = \frac{1}{2c} A v_3 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

To fid u_4, we need to solve

$$\begin{bmatrix} a & a & a & a \\ -b & -b & b & b \\ -c & c & -c & c \end{bmatrix} w = 0.$$

This results in

$$w = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

and hence we should take

$$u_4 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Finally, we have the following singular value decomposition:

$$\begin{bmatrix} a & -b & -c \\ a & -b & c \\ a & b & -c \\ a & b & c \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(5b): The best rank 2 approximation can be found as follows:

$$\frac{1}{2}\begin{bmatrix}1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2a & 0 & 0\\ 0 & 2b & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -1 & -1 & 1\\ 1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}2a & 0 & 0\\ 0 & 2b & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}a & -b & 0\\ a & -b & 0\\ a & b & 0\\ a & b & 0\end{bmatrix}.$$

Consider the matrix

$$M = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

a. Find eigenvalues of M.

b. Is M diagonalizable? Why?

c. Put M into the Jordan canonical form.

REQUIRED KNOWLEDGE: **eigenvalues/vectors, diagonalization, Jordan canonical form**

SOLUTION:

(6a): Charateristic polynomial of M can be found as

$$\begin{aligned}\det(M-\lambda) &= \det\left(\begin{bmatrix} 2-\lambda & 1 & 0 \\ -1 & 1-\lambda & 1 \\ 1 & 0 & -\lambda \end{bmatrix}\right) \\ &= -\lambda(\lambda-2)(\lambda-1) + 1 - \lambda \\ &= -\lambda(\lambda^2 - 3\lambda + 2) - \lambda + 1 \\ &= -\lambda^3 + 3\lambda^2 - 3\lambda + 1 = -(\lambda-1)^3.\end{aligned}$$

Therefore, M has the eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1$.

(6b): The matrix M is diagonalizable if and only if it has 3 linearly independent eigenvectors. To find the eigenvectors, we need to solve the equation $(M - I)x = 0$. Note that the system of equations

$$(M-I)x = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} x = 0$$

is equivalent to that of

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x = 0.$$

Therefore, the general solution is of the form

$$x = \begin{bmatrix} a \\ -a \\ a \end{bmatrix}$$

where a is a scalar. This means that we can find only one linearly dependent eigenvector for the zero eigenvalue. Consequently, M is not diagonalizable.

(6c): Since there is only one linearly independent eigenvector, Jordan canonical form consists of one block. Note that

$$(M-I)^2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad (M-I)^3 = 0.$$

Next, we solve

$$(M-I)^2 v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

One possible solution is

$$v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Note that

$$(M - I)v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Let

$$T = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

and note that

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{J}.$$
